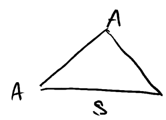
Solving Triangles: The Law of Sines & The Ambiguous Case

In the prior worksheet, you developed an understanding of how to work with the Law of Sines to solve non-right triangles. This law can be implemented when you are given two angles and any side (in other words, when you are given ASA or AAS).



Two sides but only 1 angle

1. *SSA (The Ambiguous Case).* As we've seen in the past, SSA cannot be used to show that two triangles are congruent (unless the pair of congruent angles are right angles).

   It is possible two triangles exist, one triangle exists, or, in fact, no triangles exist. The SSA scenario is therefore considered to be the *ambiguous* case.

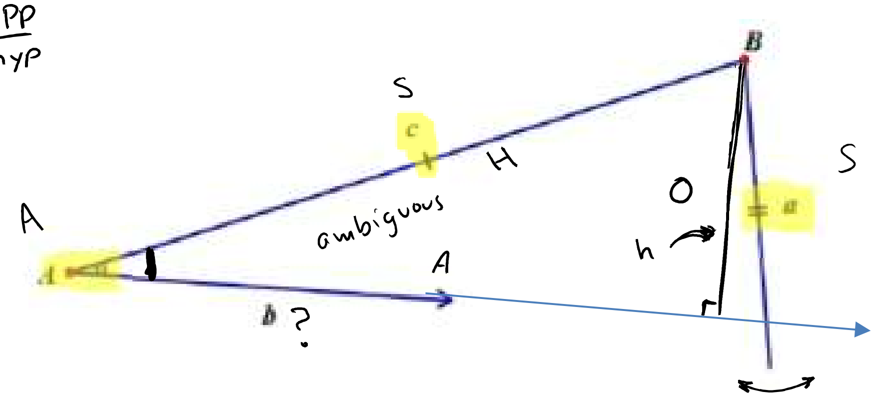
   Trigonometry can be used to help us work with this "ambiguous case". We can use what we know about basic right-triangle trigonometry to determine whether a SSA scenario results in two triangles, one triangle, or no triangle. Let's see how to do this!

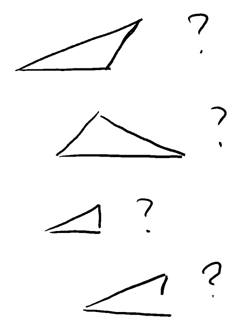
   The following hypothetical picture is convenient to use when thinking about SSA. The given SSA information is *angle A, side c*, and *side a* (take a second here to make sure you see that this is indeed a SSA case in the diagram below). Take particular note of the fact that *side b* is not given, but its direction is fixed.

Altitude = $\frac{opp}{hyp}$

$\sin A = \frac{a}{c}$

$c \sin A = a$





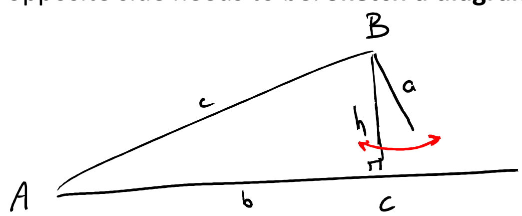
   We will focus on **side a**. The central concern at hand is the following: is **side a** long enough to intersect *side b*, considering that the angle between *sides c* and *a* is not set AND the length of *side b* has yet to be determined? If side a is long enough, the next question is: how many triangles are now possible?

   To be able to tackle this, let's consider the following. We will let *h* represent the shortest distance from point B to the opposite side of the triangle, *side b*. Take a moment to make sure you understand h can be obtained using side c and angle A in the following way, *h = c sin A*. Sketch in *h*.

2. SSA possible cases: 0/1/2 Triangles

a. Case 1 – No Triangle Possible

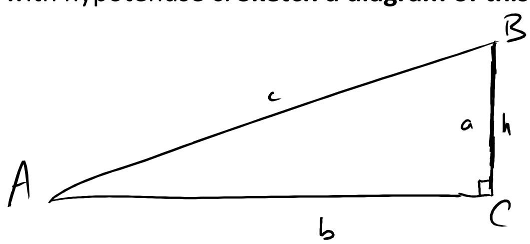
If $a < h$ (where $h = c \sin A$), then no triangles can be formed since a cannot reach where the opposite side needs to be. **Sketch a diagram of this scenario**:



No triangle can be formed when $a < h$!

b. Case 2 – One (right) Triangle Possible

If $a = h$ (again, where $h = c \sin A$), then <u>exactly one and only one triangle</u> can be formed, a right triangle with hypotenuse c. **Sketch a diagram of this scenario**:

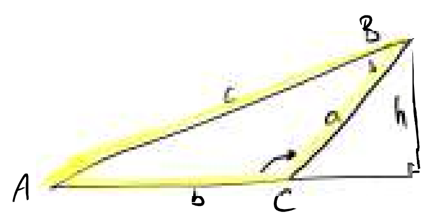


$a = h$ $\therefore \angle C = 90^\circ$ and 1 right triangle is possible

c. Case 3 – Two Triangles Possible

If $h<a<c$ (again, where $h = c \sin A$), then a can either lie to the left of h or to the right of h and still form a triangle – therefore we have <u>two options as to the shape of this triangle</u>. Note that *a* still has to be less than the length of c or else one of the triangles cannot be formed. **Sketch a diagram for EACH of the two triangle scenarios**:

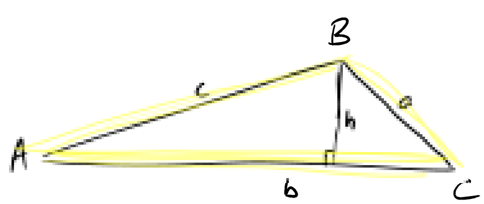
Scenario 1



$a > h$ & $\therefore \angle B$ is acute

$\angle C$ is obtuse

Scenario 2

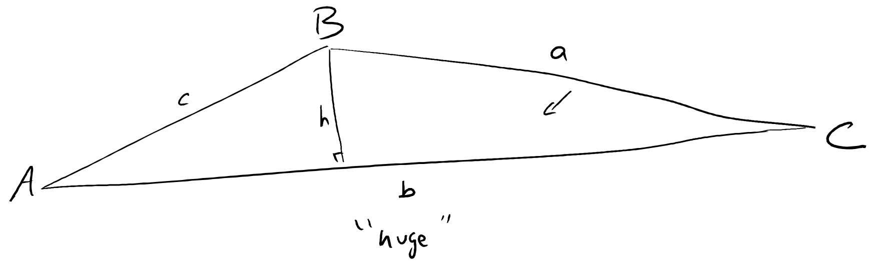


$a > h$ & $\therefore \angle B$ is obtuse

$\angle C$ is acute

d. Case 4 – One (non-right) Triangle

If a ≥ c, then exactly one and only one triangle can always be formed. **Sketch a diagram of this scenario**:



∠ B is obtuse

Summary: Solving Triangles in the Ambiguous Case. If you are given SSA in a triangle, use the four cases to determine how many triangles you will be solving.

- If only one triangle exists (either one of two scenarios), use the Law of Sines to solve the triangle.

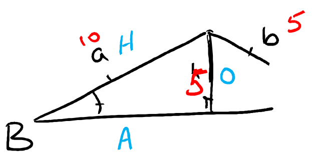
- If it two triangles are possible, sketch each triangle separately and use the *Law of Sines* to solve each triangle (you will use a result from one triangle to help you work with the second triangle).

- If no triangles are possible, obviously you have nothing to solve for and you are done!

NOTE: be very careful when you sketch your diagrams to **<u>ONLY</u>** use the given information! Take good care in sketching a *reasonable* triangle, given the measures of the provided parts of a triangle

Eg 1 Given $\triangle ABC$ $a = 10$ $b = 5$ $\angle B = 30°$ SOLVE AMBIGUOUS

S S A



① Find the altitude

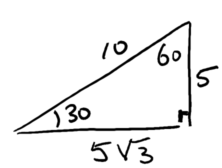
$\sin B = \frac{h}{a}$

$a \sin B = h$

$10 \sin 30 = h$

$10(0.5) = h$

$5 = h$



$a^2 + b^2 = c^2$

$5^2 + b^2 = 10^2$

$25 + b^2 = 100$

$b^2 = \sqrt{75}$

$b = 5\sqrt{3}$

$b = 5$ & $h = 5$ $\therefore$ 1 $\triangle$

$b = h$!! 90° triangle